Mathematics For Measurement: "Math for practical arts"

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Abstract

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This one-semester course resembles the usual "math for liberal arts" course in not requiring college-level prerequisites and in not being intended as part of a sequence, but differs by its selection of topics to support a theme of measurement and by its agenda of showing the utility of mathematics in expressing practical knowledge and in aiding communication by artisans with engineers. The course is a collaboration between a math/statistics teacher and a person experienced in industrial measurement processes, who together have written the learning materials now used.

Topics include functional modeling, practical trigonometry, noise description, error sensitivity and propagation, and calibration. Students learn to use Excel spreadsheets in extensive numerical, formulaic, and graphical investigations; they also learn to use the direct and inverse trig functions of scientific calculators. Linear functions are visited from multiple perspectives, including local slope of nonlinear functions and calibration for measurement bias/scale errors. A mix of graphical, numerical, and algebraic representations and methods are used, including examples of each approach which enable solution of problems that are more difficult with the other methods.

Introduction

MOTIVATION FOR DEVELOPING THE COURSE

The wider objective for the Mathematics For Measurement course (henceforth MFM) is to acquaint its students with the power of mathematical approaches in aiding analysis, description, and performance of the practical tasks that are encountered by skilled workers in all fields. The purpose is thus not training in useful vocational techniques (although practical utility is used to attract and hold student attention) or preparation for further mathematical studies (although the course may well raise that prospect in the minds of students to whom it would not otherwise have occurred). The main purpose is *to make connections between mathematical thinking and the sophisticated practical thinking of which students are already capable*. It is felt that such connections are essential to developing the flexible problem-solving capabilities that are required to deal successfully with the dynamic work situations that contemporary students will encounter in almost all fields.

The urgency of this goal stems from the deep alienation from mathematics that the majority of students feel by the time they enter college. That this is not just a result of student immaturity or intellectual incapacity is indicated by the fact that this alienation is shared by most successful adults (including most college teachers, except in the minority of disciplines that use advanced mathematics). Even among people who perform well in mathematics courses in college (including the calculus course that is commonly required for a much wider range of majors than would seem indicated by the content of those fields), only a small minority retain competence or interest in mathematics a decade later.

One source of this alienation is the approach typical of scholastic mathematics instruction, in which a progression of new techniques are taught at each level but applied to simple or abstract problems that do little to develop a student's ability to think mathematically about the world. The standard school sequence is *simplistic applications of increasingly sophisticated techniques*, rather than the *increasingly sophisticated application of simple techniques* that would be much more effective at promoting intellectual growth and in sharpening the pattern-perception skills that underlie mathematical thinking. (Some math courses not on the main sequence, notably elementary statistics and the new functions-and-modeling courses, avoid this problem to some extent, as do many advanced science courses.)

MFM is intended to provide an example of a course that uses an "increasingly sophisticated applications of simple techniques" approach to promoting mathematical insight, and can be used successfully with a broader range of students than existing courses. Note that the sophistication is largely drawn from concepts (e.g., measurement-process stability) that students have already developed in practical contexts – MFM is a course designed for adults, people who have already developed coherent sets of concepts and strategies in many areas. This presents a different set of opportunities and limitations than exist for younger students.

DIFFERENCES BETWEEN LIBERAL-ARTS AND PRACTICAL-ARTS STUDENTS

A standard educational tactic for addressing the mathematics alienation of college-level students whose interests are in non-mathematical fields is the "mathematics for liberal arts" course, which selects some topics of social or esthetic interest (e.g., congressional apportionment, fractals) and examines them from a mathematical perspective. Such a course typically has no college-level mathematical prerequisites and has no successor course depending on specific accomplishments. While these conditions leave the rigor of such courses subject to suspicion (since they lack the discipline that ruthlessly reveals shortcomings or gaps in courses imbedded in a sequence), the course has proven valuable and is widespread in American colleges.

However, the standard math-for-liberal-arts course formats have serious drawbacks for students in those college majors that are oriented toward preparation of students for particular areas of technical, but generally non-mathematical, practice. Such students, who might be called *practical arts* majors, form a significant minority of the enrollment of American community colleges. Examples of such majors (all represented in the MFM classes completed so far) include health sciences such as nursing, construction arts such as welding, and mechanical arts such as automotive mechanics.

In general, practical-arts students differ from liberal-arts ones in several ways that must be taken into account in course design if instructional effectiveness is to be maximized. Their learning goals are usually more focused, and their attention to individual topics is more dependent on the perceived relevance of each topic to those goals. There is often an active resistance to abstract generalizations, which are seen as tools of an unrealistic scholastic culture that, for example, values grammar/spelling more than effectiveness of expression and numerical exactness more than how well the source of the number corresponds to the concept it purports to quantify. Although the criticism of schools leveled by such an analysis may be well justified as an overall judgment, the attitude it engenders poses severe challenges to even the best instruction in mathematics, for which abstraction, generalization, and exact statement are central techniques.

On the other hand, it is also common for practical-arts students to have substantial practical vocational experience, which fosters familiarity with "rule of thumb" heuristics (some with an underlying mathematical basis) that are part of the lore of their trade, as well as skill at detecting the oversimplifications that are common in what are typically presented as "applied" problems in mathematics courses. This kind of sophistication makes establishing practical relevance for a math course's problems simultaneously more *difficult* (because inauthenticities will not pass unnoticed and authentic domain-specific problems may exceed the scope of the course or the reach of most of the mixed set of students) and more *important* (because mathematical thinking will be much more likely to take root if it can be connected with already-valued professional thinking patterns).

GOAL: A "MATH FOR PRACTICAL ARTS" COURSE

At one time, it was common for American school curricula to include advanced arithmetic topics that served as the "sophisticated application of simple techniques" that are now missing from most mathematics courses. For example, a problem might be set to compute the economic feasibility of a road from the cost of the materials, the longevity and repair costs for road compositions of various mixes, the impact of the road on the value of the property it serves, and the tax rates that could be assessed to fund it. All the computations involved were simple, but solution of the problem depended on sustained quantitative thinking about matters that had real-world significance, rather than on recognition of a problem type and unreflective application of its associated memorized solution template. The Mathematics For Measurement course is an attempt to recapture this spirit while also enhancing student competence with important tools (e.g., spreadsheet programs, technical drawing), concepts (e.g., noise propagation, error sensitivity), and techniques (e.g., curve-fitting, practical trigonometry).

The choice of *measurement* as the unifying theme for a "math for practical arts" course is an attempt to address the difficulties discussed above. Measurement is an activity with which almost all adults will have had some authentic experience and which is relevant in some form, often with subtle refinements, to almost all vocational areas. It is connected to several mathematical areas that are accessible with modest prerequisite skills, are of immediate utility both in practical work and as mathematical tools, and are related to advanced topics that can be identified for potential further learning but not included as part of the course. Measurement problems are also well suited for investigation with spreadsheet programs, whose mastery will give students a robust tool of lifelong utility for investigating and communicating numerical data.

MFM course topics and objectives

PRINCIPLES GUIDING MFM TOPIC SELECTION AND INSTRUCTIONAL TACTICS

One of the analytical viewpoints found most useful in assessing possible topics for the Math For Measurement course is described by Seymour Papert's "constructionist" principles (Papert, 1980, p. 54) for standards that mathematics topics should meet:

Continuity Principle: *The mathematics must be continuous with well-established personal knowledge from which it can inherit a sense of warmth and value as well as "cognitive" competence.*

Power Principle: *It must empower the learner to perform personally meaningful projects that could not be done without it.*

Principle of Cultural Resonance: *The topic must make sense in terms of a larger social context.*

These principles provide guidance for connecting to a student's past, present, and future, respectively. For MFM students, the "well-established personal knowledge" will be based in practical measurement experiences (mostly derived from work situations), the "personally meaningful projects" will be related to either current work or to increased capability in related vocational courses, and the "larger social context" to be considered is the network of activities (scientific and cultural as well as directly occupational) that support and/or use the work of vocational sectors with which students are involved, especially communication about quantitative matters.

An implication of these principles is that each mathematical topic introduced should be well anchored in concrete experience. This is not simply a matter of careful selection of authentic examples — the techniques themselves need to emphasize connections to graphical and numerical methods that can be used as robust checks of any formulaic methods. In this context, the goal is to foster generalization (seeing how patterns recur, and can be treated similarly, in various situations), not abstraction (studying the pattern of a situation to the exclusion of the meaning of the components that form the pattern).

Concreteness in the MFM course is not a matter for apology, and does not consist simply of translating mathematics into less-powerful domain-specific language. There is also a flow of sophistication back into the mathematics from the practical sectors. The idea of an approximate number is just as mathematical as that of a transcendental one (and more sophisticated in some ways). The signal/noise distinction also poses mathematical questions of substantial depth. The challenge in course design has been to respect and make use of such practice-oriented subtleties while keeping student experiences focused on mastering a toolkit of modest size and clear immediate relevance.

The desire for a one-semester, low-prerequisite, high-external-relevance course whose goal is to build mathematical sophistication in several areas places severe constraints on the size and mathematical sophistication of the set of topics that can be included. On the other hand, in many cases the omission of material present in the standard courses is seen as a positive advantage, enhancing the conceptual clarity of the MFM course and promoting authentic mathematical experiences, rather than being simply a side effect of a limited time budget. Topics were selected for the course for their perceived potential for supporting sophisticated practical thinking of the kind typical of engineering practice, rather than for their mathematical interest *per se*.

PREREQUISITES AND OBJECTIVES FOR THE MFM COURSE

The prerequisite mathematics-related skills for the topics covered in the MFM course are modest – basically through the first year of high-school algebra. Scientific calculators are used, but prior familiarity with the transcendental functions supported by them, or with the exponential notation that such calculators sometimes use, is not required. Spreadsheets are also used extensively, but familiarity with them is not required (and is unusual among current students).

The instructional materials for each topic are prefaced by a list of instructional objectives for the students. These can be examined at the MFM web site. The higher-level objectives for the course can be summarized as:

[1] Enable students to state much of their existing quantitative knowledge about their areas of specialization in terms of numeric values and appropriate formulaic relationships, and give them confidence in the utility and relevance of such statements.

[2] Enhance the ability of students to communicate on measurement-related matters with engineers or similar mathematically-adept leaders of their communities of vocational practice.

[3] Provide, for each area of mathematical technique learned, concrete methods that can be used to check, illustrate, and approximate it (such as graphs for functional formulas and scale diagrams for trigonometric problems).

Because of the stand-alone nature of the MFM course, there is substantial flexibility in the choice of specific instructional objectives to support the overall goal of promoting connections between mathematical and practical thinking. While several topics follow naturally from the selection of a theme of measurement, the balance and arrangement of the topics, as well as the particular instructional tactics, has been modified over the development of the course. Students are provided the MFM instructional materials in topic modules, of which the current 25 are listed in the next section.

The topics are arranged to support a "spiral" instructional approach, with each of the main MFM subthemes (slope, approximation, noise, curve-fitting, trigonometry) revisited with increasing sophistication. This variety is also intended as part of an effort to encourage students to approach the course as the development of a set of general analytical tools, rather than as a sequence of specialized topics.

TOPIC MODULES FOR THE MFM COURSE (AS OF SPRING SEMESTER 2006)

Part 1. (4 weeks) – Review and Basic Tools

Algebra Review - Solving Equations and Evaluating Expressions
Rounding
Using a Calculator
Formulas - Computing and Graphing
Using a Spreadsheet
Angles and Construction of Diagrams
Linear Equations - Algebra
Linear Models - Word Problems
Introduction to Data and Modeling
Propagation of Errors due to Rounding

Part 2: (3 weeks) – Basic Trigonometry & Use of Approximations

Introduction to Trigonometry
Trigonometric Ratios and Relationships
Computing with Approximate Numbers: Significant Digits
Measurement Sensitivity: Sensitivity of a Formula to Errors in Input Values
Communicating the Results of Computing with Approximate Numbers

Part 3: (5 weeks) – Measurement Noise & General-Triangle Trigonometry

Curve Fitting: Separating "Signal" from "Noise"
Describing Noise in Measured Values: Standard Deviation
Propagation of Noise I: One Measured Input Value into a Formula.
Sine and Cosine Formulas on Larger Intervals
Solving General Triangles
The "Ambiguous Case"

Part 4: (4 weeks) – Correction and Combination of Measurements

Removing Bias from a Measurement Process: Calibration
Propagation of Noise, Part II: Averaging Multiple Measurements – A Useful Rule
Propagation of Noise, Part III: Combining Measured Input Values – Empirical Method
Propagation of Noise, Part IV: Combining Measured Input Values – Other Rules

Solving Applications Problems (students select project problems from teacher-supplied list)

CONTRAST TO RELATED EXISTING COURSES

The development of the course objectives, and of the tactics used to achieve them, came mainly from critical analysis (for relevance to the goals and target audience of this course) of each of several standard mathematics and science courses, as well as domain-specific vocational courses. A compressed recapitulation of that analysis in a later section discusses how the MFM course is *not* like standard math courses (e.g., statistics, modeling) that touch on some of the same topics, and why the needed result cannot be as well obtained via non-mathematics courses such as science labs, pre-engineering courses, or vocation-specific courses.

Contrast to a functions-and-modeling course

The Mathematics For Measurement course has many common elements with the new “functions and modeling” courses that are being developed and offered at both the secondary and college levels. Such courses have great potential to promote the expansion of mathematical thinking and perception that is the goal of the MFM course. Modeling is covered in MFM classes from several perspectives, especially in fitting formulae to data using spreadsheets.

The differentiating factors of the MFM course are [1] the lower demands made on mathematical (especially algebraic) dexterity, as reflected by the limited variety of functional forms addressed, [2] the greater demands made on non-mathematical experience, as reflected in its recurrent appeal to external practical situations that can be productively examined from a mathematical perspective, and [3] the use of a central theme, measurement, that both anchors the course in non-mathematical experience and provides a context for connecting a variety of distinct strands of mathematical thought (functions, statistics, trigonometry, and approximation) both to each other and to existing student concepts and competencies.

Contrast to a quantitative-literacy course

One standard variant of the math for liberal arts course is designed to ensure that students understand, and to some extent can use, the main quantitative techniques to which non-specialists are exposed in contemporary society. The topics are typically drawn from logic, statistics, probability, financial calculation, and discrete mathematics.

This type of course is an expression of a welcome trend toward relevance and general accessibility in mathematics education, and as such is clearly compatible with the goals of the MFM course. The main differences between the courses are that the MFM course is much more focused in its topic selection and is oriented toward the expansion of work-related capability, and toward the establishment of the relevance of mathematical thinking to such capability, rather than toward general familiarity with a broad range of quantitative techniques.

Why not integrate the material into vocation-specific courses?

One strategy used for connecting mathematics with practical knowledge is to add appropriate mathematical applications (e.g., dosage calculations for nurses) into career-specific courses. This establishes relevance and utility, and provides opportunities to familiarize students with the particular math-related methods and reference materials normally used in the profession.

Particular-application training of this kind seldom contributes to the development of mathematical thinking, however, since its emphasis is usually on following standardized procedures and applying standard formulae (whose value in part is precisely that they reduce the need to think about their area of application) rather than on analyzing a situation to extract information from it by mathematical techniques. This tendency is reinforced by the limited mathematical sophistication of many of the teachers of such courses, but its root cause is that career-specific courses already have a full agenda and cannot afford to expand on the mathematical aspects of their subject.

The premise of the Mathematics For Measurement course is that, rather than trying to add the expansion of mathematical sophistication to the goals of vocational courses, it will be more effective to provide a separate mathematics course that is designed to make useful mathematical thinking visible, and then to get the vocational teachers both to provide authentic application problems suitable for use in the MFM course and to make use of some of the techniques covered in the MFM course in advanced problems in the vocational courses.

Why not cover the material in a pre-engineering course?

Topics such as calibration, measurement-noise analysis, and process modeling are actually best characterized as engineering concepts rather than as either scientific or mathematical ones. Further, engineers generally comprise the technical leadership of the sectors in which practical-arts students will work, and command their respect as role models. This suggests that a course specifically identified with engineering might be best situated to help practical-arts students transcend their mathematics alienation.

The main difficulty with making use of this suggestion is that the few explicit pre-engineering courses that are currently offered are designed to prepare students to enter college baccalaureate engineering programs, and thus require much more math than the practical-arts students for which the MFM course was designed will have mastered (or need to master). Most secondary schools do not distinguish between preparation for scientific and engineering programs (since both need the same good math and science foundation), and the increasing emphasis on integration of application-related material into advanced math and science courses gives little motivation to change this situation.

A new "general engineering" course offered at about the level of a secondary-school general-science course (or perhaps integrated with one) would be an excellent place to deal with much of the MFM material, although it would have to be adapted to reduce dependence on existing practical sophistication. But by the college level, the educational paths of engineers and of the technicians who will support them have already diverged, with differences in mathematical-thinking mastery one of the main distinguishing factors. But the connection is still quite relevant — *an important goal of the MFM course is to enable practical-arts students to communicate productively with engineers in their area*, much more than with mathematicians.

Contrast to a standard statistics course

Statistics is the only standard mathematics course in which the issues of measurement are addressed at all, and it provides an essential basis for any extended investigation of the subject. But the emphasis of statistics courses on the description of parameters of *populations* (whose distributional forms are empirical rather than mathematical) obscures the simpler concept of *measurement of an actual value*. There is an important epistemological difference, reflected in easier accessibility to authentic student comprehension, between the concept *repeated approximate measurements of the same value* that is focused on the in the MFM course and the statistical concept of *collection of a sample from a heterogeneous population of values*. The fact that the first case can be handled mathematically as a special case of the second, while an interesting result for an advanced course (and essential to the full analysis of complex measurement situations), should be omitted from the first examination of measurement concepts.

Because measurements are presented as being somewhat noisy reports of a single actual value from a continuous range, and because almost all measurement processes are approximately normally distributed in some natural choice of parameters, the statistical issues associated with measurement sets can be greatly simplified. The simplifications used for the MFM course are listed below.

The arithmetic mean (trimmed if outliers are suspected) is taken as the best estimate of the central tendency of the measurements. The possibility of bias and the calibration methods to detect and correct for it are addressed promptly. There is no need to discuss the mode statistic, and the median can be presented as a case of extreme trimming of the mean. Since the natural concept of a "most typical value" corresponds quite well with the idea of a trimmed mean of samples from a unimodal population, this is a case of providing a mathematical label and computation method for an existing student concept, a connection that is emphasized in instruction.

The stability of a measurement process is described in the MFM course by the standard deviation of a set of measurement values. The ease of computation provided by the corresponding spreadsheet function enables students to connect this measure with the perceptions they already have for noise. Because measurement processes generally produce data distributions close to normal, there is no need to emphasize the difference between population and sampling statistics.

Even when sensitization of students to the possibility of non-normal measurement processes is desired, this goal is best approached in a course of this kind by presenting the problem of detecting whether a set of measurements has been produced by mixing values from more than one normal process with different means and/or stabilities. Consideration of such situations, which are the source of most outliers or abnormal distributions in actual measurement data, provides students with mental tools to understand the sources of variation for a measurement process of interest, rather than to merely describe its pattern of variation. The goal of the MFM course is to produce problem-solvers (or at least problem-recognizers), not recording clerks.

One way of looking at MFM's limited subset of statistical topics is as using from statistics the products of one of its three historical roots: the mathematics developed about 200 years ago to calibrate, assess, and reconcile astronomical observations. The stars are constant in their courses (on the time scales and resolutions accessible to circa-1800 technology), so the position of each star is simply a stable fact – it is the process of measurement itself that must be abstracted to explain and handle the differences in measured positions, and that abstraction readily yields a very useful set of mathematical tools. Neither the descriptive-statistics tools for general population description, inferential-statistics topics such as hypothesis testing, nor impressive theoretical results such as the central limit theorem are needed to accomplish this task, and they should not be permitted to distract attention from it.

Contrast to measurement in science laboratory courses

The other type of standard course in which measurement is often explicitly included is the laboratory portion of science courses, especially in physical sciences such as chemistry and physics. These can be extremely valuable experiences for students, and are probably currently the venue of most school-inspired illuminations about measurement, especially about the nature and characteristics of measurement bias and noise.

But the focus of most school science experiments is validation of the scientific laws that summarize simple fundamental interactions, or demonstration of effects that will serve as occasions for which scientific explanation will be provided (or evoked from the student). The lab work, especially the detail of the measurement process, is firmly subordinated to the basic goal of student mastery of the scientific concepts (e.g., molecule, reaction, force, energy) required to enable scientific thinking, and of the basic relationships between those concepts. Students are being trained to look *through* the bias and noise in their measurements (and even through the measurements themselves), not *at* them.

While science-course demonstrations of the ability of mathematical laws to summarize and predict scientific relationships are in accord with the MFM agenda of showing the potential of mathematical thinking for practical use, the typical effect of science courses on practical-arts students is more intimidation than inspiration. Because of the prestige of science and the conflict of some of its most famous results with ordinary experience, most people want to be assured that "it's not rocket science" before attempting to think about a problem – science, especially in aspects that require mathematics, is seen as too hard for ordinary people. While this opinion can and should be challenged, the MFM course offers a way to advance ordinary-person mathematical sophistication without waiting for success in the parallel efforts in science education.

There is also a widespread belief that high-culture tools such as science and mathematics can only be used in situations in which all causal factors are understood or are arranged with controlled-experiment tidiness. There is some truth to this, in the sense that naïve application of scientific analysis to practical activities seldom succeeds in fully achieving the knowledge that has been accumulated empirically by practitioners, and usually produces an oversimplified model that would be disastrous to depend on blindly. Thus the skepticism of engineering practitioners about overreaching theoretical analysis is well-based. It is not surprising that this skepticism is shared, and extended to mathematical thinking, by practical-arts students for whom such practitioners are role models.

On the other hand, the history of the last century or two has shown that the combination of empirical knowledge with mathematics and scientific theory can greatly advance understanding and capability in almost all fields, especially when the theory is used to organize and illuminate practical experience rather than to supplant or discredit it. To date, the productive interactions based on this approach have been primarily at the engineering-design level, involving a limited number of people. The MFM course is designed to make the approach more available to the people who will actually be doing the day-to-day work.

Contrast to a standard trigonometry course

The MFM course includes both practical trigonometry (setting up and solving triangles, given sufficient information) and examination of sine, cosine, and tangent (and their inverses) both as ratios and as general functions. It also covers the Pythagorean theorem and its associated $\sin^2 + \cos^2 = 1$ identity, and the simple (and easy to prove) tan = sin/cos and cos(A)=sin(90º-A) identities, but excludes any further work with identities and does not discuss formulae such as those for the trigonometric functions of sums and differences of angles. Thus the MFM course omits the algebraic manipulations and functional transformations that are among the main activities of a standard trigonometry course.

The traditional trigonometry-course emphasis on the special-case 30º/60º/90º and 45º/45º/90º triangles is also avoided, since it is considered to be counterproductive to encourage students to look for special cases rather than to learn the general technique, which calculators make easy to apply in any case. In fact, the MFM course encourages calculator use, using its trigonometry section to ensure student fluency in scientific-calculator usage, with special attention to inverse functions.

On the other hand, the MFM course contains as part of its treatment of triangles some material that would normally be handled in a geometry course. This includes a derivation of the triangle-area formula, with a simple proof that illustrates how some exact relations can be shown to be true by logical argument rather than just being demonstrated by example.

The limited appeals to logical proofs related to triangles are more than balanced by extensive use of direct physical measurements and empirical investigations via calculator-based computation of trigonometric functions for particular values. An example of the tactics used is the distribution to each student of a uniquely-shaped right triangle (cut from the corners of sheets of construction paper), permitting "parallel-processing" (and unique-answer-for-each-student) investigation of conjectures by having each student test the specific case their own triangle represents. This tactic was also designed to reinforce the real-object substrate to which the measurement process is applied, in an attempt to establish that even though MFM is a math course, it is not unwilling to use direct experience for inspiration and guidance.

TOPICS COVERED IN MFM THAT ARE NOT INCLUDED IN OTHER STANDARD COURSES

Certain topics central to mathematical thinking about measurement are not directly covered in any standard courses, although they are sometimes referred to incidentally in the context of practical problems. Thus they are not really ignored, but rather taken for granted, precisely because they reflect the kind of independently-achieved quantitative sophistication that is depended on but not acknowledged by school mathematics. A major goal of the MFM course is to enable students to consciously discuss such capabilities, and thus refine them. The main topic areas of this kind are approximate numbers and measurement theory. In addition, the MFM course makes extensive use of scale drawings and of a spreadsheet program.

Approximate numbers

A significant portion of the course is directed at increasing student sophistication about the idea of numerical expression of approximate values, including review and higher-level explanation of rounding methods and of scientific and engineering notation, as well as of the shortcomings of significant digits for accurate communication of the precision of a numerical result. The question of numerical significance is approached both from the vantage of rounded-off numbers (where the possible errors are clearly delineated) and from that of repeated measurements of some object feature, where the lack of exact repeatability indicates the limits on the amount of information a measurement produced by that process conveys about the feature measured.

Measurement theory

The measurement process itself is examined, with the introduction of systematic ways of dealing with offset bias and scale-factor bias via calibration. The characteristics of random error are also examined. The effect on expected error of combining measurements is derived for the main different methods of combination, especially the important special case of averaging. The curve-fitting capability of the spreadsheet program is used to extract a signal, and then the residual noise, from noisy data sets. The phenomenon of error sensitivity is investigated both numerically and graphically.

While advanced topics such as whether measurements are independent or correlated are not discussed theoretically in any detail, the empirical techniques taught do not assume independence. This permits them to be used confidently. In special cases (which would be investigated only by more advanced students as part of an case study selected as a project), empirical results could be contrasted with formulaic predictions for independent values to assess the extent to which measurements are independent.

Diagrams

MFM students use scale diagrams drawn with simple drafting tools (ruler, protractor, compass, and drafting triangle [for right angles]) both for exploratory illustration of problems and as a solution method that can be used either for checking or to give approximate solutions. Because students are much better able to detect blunders in diagram constructions than in numerical computations, diagram use substantially increases the robustness of student use of the associated mathematical techniques.

Spreadsheet program usage

MFM uses Excel in a variety of ways, and spreadsheet use has become one of the main unifying tactics of the course. Usage starts on the first day of the course and builds in sophistication over the semester. The introductory usage is to produce numerical values for a formula. The next use is to graph formulas and measurement datasets. The curve-fitting capability is then used to separate signal from noise in measurement data. Then the signal is subtracted from the noise, leaving a residual plot that is examined to determine the quality of the fit. When the fit is good (no pattern in the residual plot), the Excel STDEV function is used to compute the standard deviation.

The ability of spreadsheets to easily compute statistics from data sets is used to teach students how to measure the noise propagation for measurement-based computations of arbitrary complexity. This is done by using multiply-duplicated sets of measurements (differing only in their noise), applying the chosen computation to each set by an Excel formula, then examining the variation in the results column with the STDEV function. This methodology is much more robust than formulaic predictions (which must make assumptions about independence), as well as being much more accessible to use by students with limited mathematical experience.

Theoretical background for the MFM course

The areas of educational theory that have contributed most to the analysis from which the MFM course was designed are those based on the classic work of Lev Vygotsky, the related later work of Lave & Wenger and Terezinha Nunes and her collaborators, and the "constructionist" analysis of Seymour Papert.

The aspect of Vygtosky's work that was most influential was the idea that mature understanding evolves from a dialectical interaction between concepts derived from heredity or experience and those derived from instruction (Vygotsky, 1978). This process can be seen separately in the development of mathematical and practical-craft competencies, each of which is built by using culturally-transmitted ideas to organize and refine direct experience and innate capabilities (and to in turn gain meaning from them), with the resulting syntheses constituting mature viewpoints in each sector. But the particular relevance for the MFM course is that under modern conditions, *fully-mature practical understanding results from a further synthesis of the mathematical and craft ways of thinking*. While ensuring such a full synthesis for practical-arts students is beyond the ambitions of the MFM course, it is intended to facilitate communication between them and the engineers in their field who have accomplished it.

The idea that a leading role is played in learning by interactions with more-capable and more-knowledgeable people was also important, both as an explanation of the origin of many ideas held by practical-arts students and as a guide to instructional tactics. Lave & Wenger's 1991 work on "legitimate peripheral participation", especially its analysis of the mechanisms of the "communities of practice" within which craft traditions are developed and propagated, was particularly helpful relating the experience of one of the authors (HE) as a technical practitioner (and a manager of such practitioners) to the issues of MFM course design. Their analysis also raises some serious questions about the feasibility of scholastic activity for promoting learning that will be usable outside a scholastic context; these are taken as valid concerns that must be addressed, in large part by efforts to situate the MFM course as a component in a larger community of shared practices that connects mathematics, science, engineering, and practical-arts disciplines.

The prior work found to be most directly relevant to the goals of the MFM course was that of Nunes *et al* (1993) relating "street mathematics" and "school mathematics". Several elements of this multifaceted report illuminate the lack of impact that school mathematics often has on practical workers, who include independent small-business owners who successfully handle essential quantitative tasks. A point seen as particularly relevant to MFM is the finding that for practical workers reduction of mathematical problems to written form often entails a disabling loss of meaning, while spoken consideration of the problem results in fewer mistakes, even for novel applications. This highlights the possibility that abstraction may be inferior to analogy as a mode of generalization for many people and situations, especially in the robustness of its conclusions in actual workplace decision processes.

Papert's ideas about using computer programming as a concrete path to abstract thinking, and the suggestive principles he offers to guide topic selection in curriculum development, also provided guidance in the task of connecting the abstract realm of mathematics with the concrete realm of the practical arts. Although most of Papert's work focuses on the articulation of mathematics with the early natural learning that makes programming Logo turtles an accessible task for elementary-school students, his "principle of cultural resonance" also provides a useful guide to articulation with the world of adult practice. The success of MFM students in using a spreadsheet program (usually for the first time) validates Papert's analysis.

IMPLICATIONS FOR INSTRUCTION

Street learning and school learning

Since the clearly-expressed goal of both students and society for practical-arts instruction is preparation of the students for effective action in the practical domains they are entering, information about what effects school instruction has on the ability of practical workers to deal with the situations they actually encounter "on the street" is of great interest. A series of well-constructed studies by Nunes and her collaborators (1993), in which the mathematics-related methods and capabilities of schooled and unschooled Brazilian children, self-employed artisans, and microbusiness owners were compared, provides several points of interest for the MFM course.

The first point is that computation in practical situations was often done by algorithms other than the standard models taught in school. For example, repeated and/or grouped addition was often used instead of multiplication in determining the total price for a purchase of several of the same item. It seems likely that this is done not simply because of ignorance of standard procedures (even schooled people generally used the typical practical method of their field, not the school-taught one), but because using an often-used, intuitively-checkable method to handle the relatively few cases of large purchases is in fact a more robust process. This implies that the MFM course should be careful not to scorn practical methods that seem inferior to mathematical ones – a better strategy is to explain and extend them (and to be alert for non-robustness in mathematical procedures).

A second point is that people showed much more competence in solving problems orally than in written form. This was true of schooled and unschooled people. Various possible explanations were investigated, and the one best supported by the evidence was that a disabling loss of meaning occurred for most people when the problem was reduced to written form. The subsequent computation thus lost corrective feedback from the person's knowledge of the structure of the situation, leading to errors or dead ends. This seems to me to be of great relevance to the MFM course, since it touches on one of the main sources of alienation by practical-arts students, who often combine substantial informal competence with weak school-mathematics skills.

In fact, preservation of meaning was a central theme of the findings in several parts of the study. Practical workers needed to do so to solve the problems at all; they would give up if meaning was lost, but would seldom assert unreasonable answers. Students were more willing to continue, but were much more likely to produce erroneous results. The lesson for the MFM course is to support methods that help preserve meaning (such as consistently labeling numbers with the appropriate engineering units, using written narrative to connect parts of a problem, and developing student skill in producing careful diagrams reflecting each problem), and to scrupulously ensure that problems describe realistic situations.

The study also showed that many practical workers could successfully transfer a computational skill (such as solution of problems involving quality-quantity-price relations) to an isomorphic problem domain. But an essential part of the transfer was for the workers to see that the pattern of *meaning* was isomorphic – if the problem was reduced to written form and handled just as an arithmetic problem, success was much lower. This suggests that for practical-arts workers (and no doubt many others), *analogy may be a more effective mode of generalization than abstraction*. This is why graph-based modeling (and graphs in general) will so often be more successful in this context than algebraic methods.

Overall, this study reinforces several of the design principles of the MFM course. Practical workers have existing, rational quantitative skills that are based on the meaning of the situations to which they are applied. Their need is to have these skills expanded, extended, and explained – not replaced.

Stages of learning

Vygotsky (1978, 1986) distinguished between "spontaneous" concepts (which arise without explicit instruction in response to direct experience, and thus start with familiar meanings but weak connections to other concepts) and "nonspontaneous" concepts provided by instruction (which start with good connections to other concepts, since that is how they are introduced, but have only vague meanings when first used). This "spontaneous/nonspontaneous" distinction can alternatively be expressed as "everyday/scientific", or "individual/cultural"— the critical idea here is the contrast between inward and outward reliance for concept meaning.

This is not a judgmental division of concept classes into "authentic" and "imposed" (or, to take the other side of that poorly-framed argument, into "naïve" and "professional"). Instead, Vygotsky perceived the productive interaction between the two kinds of concepts as learners use them in their communication with others and in thinking about and extending their experiences. Spontaneous concepts acquire more structural connections (via abstract intermediary concepts introduced by instruction) while such nonspontaneous concepts acquire more connotative connections to immediate perceptions (via observations of the concept's effect in use, especially when it touches familiar concrete objects). Thus the two concept-development processes reinforce each other rather than compete, transforming concepts from both sources into mature integrated concepts that are well connected both to other concepts and to experience and innate capabilities.

This concept transformation is an ongoing process, with each round of synthesis adding to an individual's set of mastered concepts, which then can be used to interact[1] with more advanced culturally-supplied concepts, leading to further synthesis and mastery. As the pool of mastered concepts expands, the number of connections between concepts grows even faster, since concepts of greater connective power become accessible and earlier connective concepts can be applied to already-mastered concepts other than those by which they were first introduced.

Linkage-driven processes of this kind can be expected to exhibit nonlinear shifts in power and behavior at a critical threshold in connectedness, a systems-theory finding (Erdos, 1960) that illuminates the long-observed existence of qualitative stages in the intellectual development of children. Hall (1904), Piaget (e.g., 1952), and Vygotsky (1978) give classic education-centered observations, but the idea is also implicit in the coming-of-age rituals (e.g., Christian first communion and confirmation) long common to all cultures. That such stages also obtain in adult learning is suggested by the tradition of apprentice/journeyman/master distinctions in skilled-craft work, as well as by the abrupt increases of capability people frequently exhibit during the process of learning particular technical skills (the classic example is learning to ride a bicycle).

Although stage transitions for some sectors may occur at predictable points in the accumulation of experience, learning is seldom simply cumulative. Expert thinking and behavior patterns usually replace (or at least substantially alter) those learned as novices, rather than just being added to them, largely because the expanded web of connective concepts reveals the contradictions that the naïve ideas have with each other or with well-established related concepts. Much of the qualitative, "quantum jump" appearance of stage transitions can be explained by such feedback effects, which will often occur in cascades as concept transformation propagates until a condition without salient contradictions is reached.

The specific implication for the MFM course of these general principles is that the course's topic choices should be designed to promote the establishment and extension of a well-connected conceptual framework, so as to facilitate the mutual transformation of related practical ideas and the use of existing practical knowledge to give meaning to the mathematical ideas used to examine it. This in turn implies that topics with a broad range of practical applications (e.g., right triangles, error sensitivity) are to be preferred to those, however elegant or powerful in their sphere, that provide only a few occasions for practical use. The value of the strategy of "increasingly sophisticated use of simple techniques" is thus explained, since it is the simple techniques that provide the wide-ranging connecting threads that lay the groundwork for the intellectual stage transition that results from development of a mathematical perspective.

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